

FINITE TIME COLLAPSING OF THE KÄHLER-RICCI FLOW ON THREEFOLDS

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ABSTRACT. We show that if on a compact Kähler threefold there is a solution of the Kähler-Ricci flow which has finite time collapsing, then the manifold admits a Fano fibration. Furthermore, if there is finite time extinction then the manifold is Fano and the initial class is a positive multiple of the first Chern class.

1. INTRODUCTION

A Fano manifold is a compact complex manifold X with ample anticanonical bundle $-K_X = \Lambda^{\dim X} T^{1,0} X$. Thanks to the Calabi-Yau Theorem [49], a compact complex manifold is Fano if and only if it admits a Kähler metric with positive Ricci curvature.

A compact Kähler manifold X is said to admit a Fano fibration if there exists a surjective holomorphic map $f : X \rightarrow Y$ with connected fibers, where Y is a compact normal Kähler space with $0 \leq \dim Y < \dim X$ and such that $-K_X$ is f -ample (see Definition 2.1 below for the definition of f -ampleness). In this case the generic fiber of f is a Fano manifold of dimension $\dim X - \dim Y$.

The simplest example of a Fano fibration is when Y is a point, and X is a Fano manifold. Other simple examples are obtained by taking $X = F \times Y$ where F is a Fano manifold and Y is any compact Kähler manifold. Fano fibrations are more general than Mori fiber spaces, where one requires in addition that the relative Picard number of f be equal to 1.

It is easy to see that if X admits a Fano fibration then there exists a Kähler metric ω_0 on X such that the (unnormalized) Kähler-Ricci flow

$$(1.1) \quad \begin{cases} \frac{\partial}{\partial t} \omega(t) = -\text{Ric}(\omega(t)) \\ \omega(0) = \omega_0 \end{cases}$$

starting at ω_0 has a solution defined only for a finite time $T > 0$, and $\text{Vol}(X, \omega(t)) \rightarrow 0$ as $t \rightarrow T$ (see right after the proof of Lemma 2.2 for details). We say that the Kähler-Ricci flow has *finite time collapsing*, by which we simply mean that the total volume of the manifold approaches zero in some finite time. It is known that if this happens then necessarily the Kodaira dimension of X is $-\infty$ (see [11, Proposition 4.2]). See also

[11, 14, 15, 16, 25, 29, 35, 36, 37, 40, 39, 43, 44, 50, 51] and references therein for more results on finite time singularities of the Kähler-Ricci flow.

In this article we study the converse question, and we formulate precisely the following conjecture which has been part of the folklore of the subject:

Conjecture 1.1. *Let X^n be a compact Kähler manifold. Then there exists a Kähler metric ω_0 such that the Kähler-Ricci flow (1.1) has finite time collapsing if and only if X admits a Fano fibration $f : X \rightarrow Y$. In this case, we can write*

$$(1.2) \quad [\omega_0] = -2\pi T c_1(K_X) + f^*[\omega_Y],$$

for some Kähler metric ω_Y on Y , where T is the maximal existence time of the flow.

Since Y need not be smooth, in this statement a Kähler metric on Y is in the sense of analytic spaces (see e.g. [3, 18, 23, 48]). Also, the factor of 2π in (1.2) is due to our definition of Ricci form, locally given by $\text{Ric}(\omega) = -\sqrt{-1}\partial\bar{\partial}\log\det(g)$, so that $[\text{Ric}(\omega)] = 2\pi c_1(X)$.

Recall now that a $(1,1)$ class $[\alpha] \in H^{1,1}(X, \mathbb{R})$ is called *nef* if it lies in the closure of the Kähler cone, i.e. if it is a limit of Kähler classes. For example, if $\omega(t)$ is a solution of the Kähler-Ricci flow on X with $0 \leq t < T < \infty$, then the limiting class $[\alpha] = \lim_{t \rightarrow T} [\omega(t)]$ is always nef.

We can now state the following conjecture in analytic geometry, which does not mention the Kähler-Ricci flow, and which is in fact equivalent to Conjecture 1.1 (see Theorem 2.3):

Conjecture 1.2. *Let X^n be a compact Kähler manifold. Then X admits a Fano fibration $f : X \rightarrow Y$ if and only if there exists a closed real $(1,1)$ form α with $[\alpha] \in H^{1,1}(X, \mathbb{R})$ nef, with $\int_X \alpha^n = 0$ and with*

$$[\alpha] + \lambda c_1(X)$$

a Kähler class, for some positive real number λ . In this case, we can write

$$(1.3) \quad [\alpha] = f^*[\omega_Y],$$

for some Kähler metric ω_Y on Y .

In particular, if this conjecture holds, then $[\alpha]$ contains a smooth semi-positive representative (cf. [44, Conjecture 1], where it is conjectured that this holds even without the assumption that $\int_X \alpha^n = 0$). As explained in section 2, it is well-known that Conjecture 1.2 is true if X is projective and $[\alpha] \in H^2(X, \mathbb{Q})$, as a simple consequence of the base-point-free theorem [26], and in fact more generally when X is projective and $[\alpha]$ belongs to the real Néron-Severi group $NS_{\mathbb{R}}(X)$. So our main interest in this problem is when the manifold X is not projective, or when the class $[\alpha]$ is not in $NS_{\mathbb{R}}(X)$.

As an aside, we remark that an analogous statement as in Conjecture 1.2 should be true when the class $[\alpha]$ satisfies instead $\int_X \alpha^n > 0$. In this case there should exist a bimeromorphic morphism $f : X \rightarrow Y$ to a compact

normal Kähler space Y (of dimension n) such that (1.3) holds. This statement follows from the results in [23] if the extremal face in $\overline{NA}(X)$ of classes which intersect trivially with $[\alpha]$ is in fact a ray (see [23] for notation), but this is not the case in general.

We also consider the following related conjecture, raised by Tian [43, Conjecture 4.4] (see also [35]).

Conjecture 1.3. *Let (X^n, ω_0) be a compact Kähler manifold, let $\omega(t)$ be the solution of the Kähler-Ricci (1.1), defined on the maximal time interval $[0, T)$ with $T < \infty$. Then as $t \rightarrow T$ we have*

$$(1.4) \quad \text{diam}(X, \omega(t)) \rightarrow 0,$$

if and only if

$$(1.5) \quad [\omega_0] = \lambda c_1(X),$$

for some $\lambda > 0$.

Condition (1.4) is called *finite time extinction*, and Conjecture 1.3 states that finite time extinction happens if and only if the manifold is Fano and the initial class is a positive multiple of the first Chern class. Of course, as we will see in section 3, condition (1.4) implies that $\text{Vol}(X, \omega(t)) \rightarrow 0$ as $t \rightarrow T$, i.e. that finite time extinction implies finite time collapsing. In the setting of Conjecture 1.1, finite time extinction corresponds to the case when Y is just a point.

In general if (1.5) holds, then (1.4) holds thanks to work of Perelman (see [34]), who proved the stronger result that under the volume-normalized flow the diameter remains uniformly bounded above. If $[\omega_0] \in H^2(X, \mathbb{Q})$ (so X is projective), then this conjecture was proved by Song [35].

Our main result is the following.

Theorem 1.4. *If $n \leq 3$ then Conjectures 1.1, 1.2 and 1.3 hold.*

In fact, we will show in Section 2 that in general Conjectures 1.1 and 1.2 are equivalent (this is essentially elementary), and that Conjecture 1.1 implies Conjecture 1.3, by modifying the arguments in [35] to show that once we have a Fano fibration, if the base Y is not a point then the diameter of $(X, \omega(t))$ does not go to zero. We are then reduced to showing Conjecture 1.2. The case when $n = 2$ is not hard to deal with. For the case when $n = 3$, our main technical tool is the very recent completion of the Minimal Model Program for Kähler threefolds by Höring-Peternell [23, 24], especially their construction of Mori fiber spaces on a bimeromorphic model of a uniruled Kähler threefold [24]. Using their results, together with some extra arguments due to the fact that the Fano fibrations that we seek to construct are more general than Mori fiber spaces, in Section 3 we construct the Fano fibration in the setting of Conjecture 1.2.

In light of these results, it is desirable to study the behavior of the Kähler-Ricci flow on the total space of a Fano fibration, with initial metric satisfying (1.2). This is in general a very hard problem. When Y is a point this amounts to studying the Kähler-Ricci flow on Fano manifolds in the anti-canonical class. In general, it is expected (see [38, 43, 44]) that as $t \rightarrow T$ the evolving metrics $\omega(t)$ converge (in a suitable sense, away from the subvariety $f^{-1}(S)$ where $S \subset Y$ is the critical locus of f together with the singular set of Y) to $f^*\omega_Y$ for some Kähler metric on $Y \setminus S$. Furthermore $(X, \omega(t))$ is expected to converge in the Gromov-Hausdorff topology to the metric completion of $(Y \setminus S, \omega_Y)$, which should be homeomorphic to Y . Lastly, for any given fiber $X_y = f^{-1}(y)$, $y \in Y \setminus S$, the rescaled metrics $\frac{\omega(t)}{T-t}|_{X_y}$ should converge in a suitable sense to a (possibly singular) Kähler-Ricci soliton. These results are essentially known when Y is a point (see [10, 31, 34]), and some progress has been made in the case of projective bundles [15, 16, 37, 40], but not much more is known in general.

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2. FINITE TIME COLLAPSING

In this section we prove Conjectures 1.1 and 1.2 when $n \leq 3$.

For the moment we work in general dimension n , and will restrict to $n \leq 3$ later on. Recall the following standard definition (see e.g. [2, p.140]):

Definition 2.1. *Let $f : X \rightarrow Y$ be a holomorphic map between compact complex analytic spaces. We say that a holomorphic line bundle L on X is f -ample if there exists $\ell \geq 1$ such that if we consider the coherent sheaf $\mathcal{F} = f_*(\ell L)$ on Y , then the natural map $f^*f_*(\ell L) \rightarrow \ell L$ is surjective and defines an embedding $\Phi : X \hookrightarrow \mathbb{P}(\mathcal{F}) := \text{Proj}_Y(\text{Sym}(\mathcal{F}))$, such that $f = \pi \circ \Phi$ where $\pi : \mathbb{P}(\mathcal{F}) \rightarrow Y$ is the projection, and so that $\ell L \cong \Phi^*\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1)$.*

To start, we make the following useful observation, which is well-known in the projective case (see [22, Proposition II.7.10]). We refer the reader for example to [3, 18, 23, 48] for the definition and basic properties of Kähler metrics on compact complex analytic spaces. Unless otherwise stated, the analytic spaces that we consider are reduced, irreducible, but not necessarily normal.

Lemma 2.2. *Let $f : X \rightarrow Y$ be a surjective holomorphic map with connected fibers, where X is a compact Kähler manifold and Y is a compact normal*

Kähler space with $0 \leq \dim Y < \dim X$. Then $-K_X$ is f -ample if and only if there exists a Kähler metric ω_Y on Y such that $f^[\omega_Y] - c_1(K_X)$ is a Kähler class on X .*

Proof. Assume that $-K_X$ is f -ample. This clearly implies that the map f is projective, and this in turns implies that f is a Kähler morphism (in the sense of [18]). More precisely we can find a metric h on $-\ell K_X$ whose curvature form ω is positive definite on all the fibers, see e.g. [3, Lemma 4.19], [18, Lemma 4.4] or [48, Proposition II.1.3.1], and then it follows that given any Kähler metric ω_Y on Y there exists $A > 0$ large enough so that $Af^*\omega_Y + \omega$ is a Kähler metric on X (see again [18, Lemma 4.4] or [48, Proposition II.1.3.1]), which is in the class $f^*[A\omega_Y] - \ell c_1(K_X)$.

Conversely, if we have that $f^*[\omega_Y] - c_1(K_X)$ is a Kähler class on X , for some Kähler class $[\omega_Y]$ on Y , then $-K_X$ is f -ample. Indeed, for every fiber F of f (which may be singular) we have that $-K_X|_F$ is a holomorphic line bundle with a smooth metric with strictly positive curvature form (in the sense of analytic spaces). Grauert's version of the Kodaira embedding theorem for analytic spaces [20] (see also [4, Theorem 1.1]) implies that $-K_X|_F$ is ample, and this is equivalent to $-K_X$ being f -ample, see e.g. [4, Proposition 1.4]. \square

We can now show the easy direction of Conjecture 1.1, namely that on every Fano fibration there always exists a solution of the Kähler-Ricci flow (1.1) which collapses in finite time. Given $f : X \rightarrow Y$ a Fano fibration, by Lemma 2.2 there exists a Kähler metric ω_Y on Y such that $[\omega_0] = f^*[\omega_Y] - c_1(K_X)$ is a Kähler class on X . Then the Kähler-Ricci flow starting at any Kähler metric ω_0 in this class has a finite time singularity at time $\frac{1}{2\pi}$ (thanks to the cohomological characterization of the maximal existence time of (1.1) given in [46, 47, 45]) and the total volume of X goes to zero as time approaches $\frac{1}{2\pi}$, because the limiting class $[\omega_0] + c_1(K_X) = f^*[\omega_Y]$ satisfies $\int_X (f^*\omega_Y)^n = 0$.

We can also show the easy direction of Conjecture 1.2. Let $f : X \rightarrow Y$ be a Fano fibration, and fix ω_Y a Kähler metric on Y (in the sense of analytic spaces) as in Lemma 2.2, so that $f^*[\omega_Y] - c_1(K_X)$ is a Kähler class on X . Then $\alpha = f^*\omega_Y$ is a smooth nonnegative real $(1, 1)$ form on X , and so its cohomology class is nef (it is the limit of the Kähler classes $[\alpha + \varepsilon\omega_X]$ as $\varepsilon \downarrow 0$, where ω_X is any Kähler metric on X), and satisfies $\int_X \alpha^n = 0$ and $[\alpha] + c_1(X)$ is a Kähler class, as required.

Next we show:

Theorem 2.3. *Conjectures 1.1 and 1.2 are equivalent.*

Proof. Since we have just shown that the easy directions of both conjectures always hold, it is enough to show that the other directions are equivalent. Assume first that Conjecture 1.1 holds, and let $[\alpha] \in H^{1,1}(X, \mathbb{R})$ be a nef class with $\int_X \alpha^n = 0$ and with $[\alpha] + \lambda c_1(X)$ Kähler for some $\lambda > 0$. Fix a Kähler metric ω_0 in this class, and consider its evolution by the Kähler-Ricci

flow (1.1). The class of the evolved metric $\omega(t)$ is

$$[\omega(t)] = [\omega_0] - 2\pi t c_1(X) = [\alpha] + (\lambda - 2\pi t) c_1(X) = \left(1 - \frac{2\pi t}{\lambda}\right) [\omega_0] + \frac{2\pi t}{\lambda} [\alpha].$$

For $0 \leq t < \frac{\lambda}{2\pi}$ this is a sum of a Kähler class and a nef class, and so it is Kähler, while for $t = \frac{\lambda}{2\pi}$ this equals $[\alpha]$ which is nef but not Kähler since $\int_X \alpha^n = 0$. Then the cohomological characterization of the maximal existence time T of (1.1) given in [46, 47, 45] shows that $T = \frac{\lambda}{2\pi} < \infty$ and the limiting class is $[\alpha]$. Therefore the Kähler-Ricci flow $\omega(t)$ has finite time collapsing, and by Conjecture 1.1, X admits a Fano fibration. Also by (1.2) we can write $[\omega_0] = -\lambda c_1(K_X) + f^*[\omega_Y]$, for some Kähler metric ω_Y on Y , and so $[\alpha] = f^*[\omega_Y]$, i.e. (1.3) holds.

Assume conversely that Conjecture 1.2 holds, and let ω_0 be a Kähler metric on X such that the Kähler-Ricci flow (1.1) has finite time collapsing at time $T < \infty$. The limiting class of the flow is

$$[\alpha] = [\omega_0] - 2\pi T c_1(X),$$

which is nef, satisfies $\int_X \alpha^n = 0$, and $[\alpha] + 2\pi T c_1(X)$ is Kähler. Therefore Conjecture 1.2 gives us a Fano fibration $f : X \rightarrow Y$, and $[\alpha] = f^*[\omega_Y]$ for some Kähler metric ω_Y on Y , which shows (1.2). \square

So we are left to consider the converse implication in Conjecture 1.2, and so we assume we have a nef class $[\alpha]$ with $\int_X \alpha^n = 0$ and with $[\alpha] + \lambda c_1(X)$ Kähler for some $\lambda > 0$. Recall that a $(1, 1)$ class is called pseudoeffective if it contains a closed positive current, and that a holomorphic line bundle is called pseudoeffective if its first Chern class is (see e.g. [12]). We have the following simple remark (cf. [11, Proposition 4.2]).

Lemma 2.4. *If X^n is a compact Kähler manifold which has a nef class $[\alpha]$ with $\int_X \alpha^n = 0$ and with $[\alpha] + \lambda c_1(X)$ Kähler for some $\lambda > 0$. Then K_X is not pseudoeffective, and therefore $\kappa(X) = -\infty$.*

Proof. If K_X is pseudoeffective, then so is the class $-\lambda c_1(X)$. The class $[\alpha]$ is therefore the sum of a Kähler class and a pseudoeffective class, and so it is big (in the sense that it contains a Kähler current [12]). But a nef and big class always has $\int_X \alpha^n > 0$, by [5, Theorems 4.1 and 4.7], and this contradicts our assumption. Also, in general K_X not pseudoeffective implies $\kappa(X) = -\infty$, since if $\kappa(X) \geq 0$ then some power $K_X^{\otimes \ell}$ ($\ell \geq 1$) is effective, and so K_X is pseudoeffective. \square

A well-known conjecture says that if K_X is not pseudoeffective, then X is uniruled (i.e. covered by rational curves). This is proved in [6] in the projective case, and is also known in the Kähler case if $n \leq 3$ by [7].

If we assume now that X is projective and $[\alpha] \in H^2(X, \mathbb{Q})$, so there is an integer $m \geq 1$ such that $[m\alpha] = c_1(L)$ for some holomorphic line bundle L , then it is well-known that Conjecture 1.2 holds. Indeed, by assumption we have that $[\alpha] + \lambda c_1(X)$ is a Kähler class, and since this is an open condition

we may assume that $\lambda = \frac{p}{q} > 0$ is positive and rational. Up to increasing m we may assume that $m\lambda \geq 1$ is an integer, and we fix ω_0 a Kähler metric in the class $[m\alpha] + m\lambda c_1(X)$. Then we have

$$c_1(L - K_X) = [\omega_0] + (m\lambda - 1)c_1(K_X) = \frac{1}{m\lambda}[\omega_0] + \left(1 - \frac{1}{m\lambda}\right)c_1(L),$$

is a Kähler class (since it is sum of a Kähler and a nef class), i.e. $L - K_X$ is ample. The base-point-free theorem (see [26]) then shows that kL is base-point-free for some $k \geq 1$, and so it induces a holomorphic map $f : X \rightarrow Y$ onto a normal projective variety Y , so that f has connected fibers and kL is linearly equivalent to the pullback of an ample divisor under f . This implies that $[\alpha] = f^*[\omega_Y]$ for some Kähler metric ω_Y on Y , and since we have that $\int_X \alpha^n = 0$, this implies that $0 \leq \dim Y < \dim X$. By construction, a multiple of $-K_X$ is linearly equivalent to the sum of an ample line bundle and the pullback of a line bundle from Y , and so f is a Fano fibration and (1.3) holds.

An extension of this argument deals with the more general case when $[\alpha]$ belongs to the real Néron-Severi group

$$NS_{\mathbb{R}}(X) = (H^{1,1}(X, \mathbb{R}) \cap H^2(X, \mathbb{Q})) \otimes \mathbb{R}.$$

This is identified with the space of \mathbb{R} -divisors modulo numerical equivalence, and is a subspace of $H^{1,1}(X, \mathbb{R})$, in general of smaller dimension.

Proposition 2.5. *Conjecture 1.2 holds if X is projective and $[\alpha] \in NS_{\mathbb{R}}(X)$. In particular, Conjecture 1.2 holds if*

$$H^{2,0}(X) = 0.$$

Proof. By assumption we have $\alpha = c_1(D)$ where D is a nef \mathbb{R} -divisor, and $c_1(D - \lambda K_X)$ is a Kähler class for some positive real number λ , i.e. $A := D - \lambda K_X$ is an ample \mathbb{R} -divisor. Then the base-point free theorem for \mathbb{R} -divisors [21, Theorem 7.1] shows that D is semiample, in the sense that there exists a surjective morphism $f : X \rightarrow Y$ to a normal projective variety Y , with connected fibers, and such that D is \mathbb{R} -linearly equivalent to f^*H where H is an ample \mathbb{R} -divisor on Y . Therefore $[\alpha]$ is the pullback of a Kähler class on Y , and as before this implies that $\dim Y < \dim X$. We also have that $-K_X$ is \mathbb{R} -linearly equivalent to $\frac{1}{\lambda}(A - f^*H)$, and so $-K_X$ is f -ample.

For the last item, it is enough to remark that $H^{2,0}(X) = 0$ implies that X is projective (an old result of Kodaira), and that $H^{1,1}(X, \mathbb{R}) = NS_{\mathbb{R}}(X)$ (thanks to the Hodge decomposition on $H^2(X, \mathbb{C})$). \square

Corollary 2.6. *Conjecture 1.2 holds when $n = 2$.*

Proof. Indeed we have $\kappa(X) = -\infty$ by Lemma 2.4, and when $n = 2$ this implies that $H^0(X, K_X) = H^{2,0}(X) = 0$, and we conclude by Proposition 2.5. \square

Finally we deal with the case when $n = 3$.

Theorem 2.7. *Conjecture 1.2 holds when $n = 3$.*

Proof. We have a nef class $[\alpha]$ with $\int_X \alpha^3 = 0$ and with $[\alpha] + \lambda c_1(X) = [\omega_0]$ for some Kähler metric ω_0 and some $\lambda > 0$. Recall that K_X is not pseudoeffective, by Lemma 2.4. Thanks to the main theorem of [7] it follows that X is uniruled, i.e. covered by rational curves. Let $X \dashrightarrow B$ be the MRC fibration of X (constructed in [8], see also [9, 27]), which satisfies $\dim B < 3$ because X is uniruled.

If $\dim B = 0$ then X is rationally connected (and since it is Kähler, it must be projective thanks to [8, Corollaire, p.212]), and so $H^{2,0}(X) = 0$ and Conjecture 1.2 holds thanks to Proposition 2.5. If $\dim B = 1$ then $X \rightarrow B$ is holomorphic everywhere, B is a compact Riemann surface of genus at least 1, and the general fiber F is rationally connected (and hence again projective). Therefore we have $H^{1,0}(F) = H^{2,0}(F) = 0$ and this easily implies that $H^{2,0}(X) = 0$ and so Conjecture 1.2 holds in this case as well.

Therefore we can assume that $\dim B = 2$, and the general fiber F of the MRC fibration is isomorphic to \mathbb{CP}^1 . Following [24], we call a Kähler class $[\omega]$ on X normalized if $\int_F \omega = 2$. We will apply the results of Höring-Peternell in [23, 24], and we are grateful to Andreas Höring for his help with the following arguments.

We consider the Kähler class $[\omega] = \frac{1}{\lambda}[\omega_0]$, so that $c_1(K_X) + [\omega] = \frac{1}{\lambda}[\alpha]$ is a nef class. Let

$$\mu = \frac{2}{\int_F \omega} > 0,$$

so that $\mu[\omega]$ is normalized. By adjunction we have $\int_F c_1(K_X) = -2$, and since $c_1(K_X) + [\omega]$ is nef we also have

$$\int_F (c_1(K_X) + \omega) \geq 0,$$

and so $\mu \leq 1$.

Assume that $[\omega]$ is not normalized, i.e. that $\mu < 1$. Thanks to [24, Lemma 3.3] the class $c_1(K_X) + \mu[\omega]$ is pseudoeffective, and so $c_1(K_X) + [\omega] = (c_1(K_X) + \mu[\omega]) + (1 - \mu)[\omega]$ is big. Since this equals $[\alpha]$, we get a contradiction to the fact that $\int_X \alpha^3 = 0$ (using again [5, Theorems 4.1 and 4.7] as in Lemma 2.4).

Therefore $[\omega]$ is normalized, and we can then apply [24, Theorem 1.4] and obtain a holomorphic map $f : X \rightarrow Y$ onto a normal compact complex surface Y , such that f has connected fibers and a curve $C \subset X$ satisfies $f(C)$ is a point if and only if $\int_C (c_1(K_X) + \omega) = 0$. Therefore $-K_X$ is f -nef and its restriction to a generic fiber of f is big. Relative Kawamata-Viehweg vanishing for complex spaces [1, Theorem 2.1] (cf. [30]) implies that $R^i f_* \mathcal{O}_X \cong \mathcal{O}_Y$ for all $i > 0$, and then [28, Theorem 1] gives that Y has at worst rational singularities (the proof there uses Grothendieck duality and Grauert-Riemenschneider vanishing, but these have been extended to the analytic setting in [32, 33] and [42] respectively). This in turn implies that Y is a Kähler space, for example by [19].

From the construction of f in the proof of [24, Theorem 1.4], we obtain a commutative diagram

$$(2.1) \quad \begin{array}{ccc} \Gamma & \xrightarrow{q} & Z \\ p \downarrow & & \downarrow \nu \\ X & \xrightarrow{f} & Y \end{array}$$

where Γ is a compact Kähler manifold (in [24] Γ is just a compact analytic space in class \mathcal{C} , but we may replace it with a resolution of singularities), p is a modification, Z is a smooth Kähler surface, and ν is the contraction of an effective divisor $E \subset Z$, which is the null locus of a nef and big $(1, 1)$ class $[\beta]$ on Z , such that $q^*[\beta] = p^*[\alpha]$.

Since Y has rational singularities, it follows from [23, Lemma 3.3] that $[\beta] = \nu^*[\gamma]$ for a $(1, 1)$ class $[\gamma]$ on Y (see e.g. [23] for the definition of $(1, 1)$ classes on normal analytic spaces). In fact, $[\gamma]$ is a Kähler class (in the sense of analytic spaces). To see this, let K be a Kähler current on Z in the class $[\beta]$ which is singular only along the null locus of $[\beta]$, which exists thanks to [11, Theorem 1.1] (the proof there simplifies vastly in the case at hand, since $\dim Z = 2$). Then the pushforward current ν_*K has local potentials everywhere on Y and belongs to the class $[\gamma]$ thanks to [23, Lemma 3.4], and it is a smooth Kähler metric away from $\nu(\text{Exc}(\nu))$, which is a finite set. It is then easy to produce a Kähler metric in the class $[\gamma]$, by modifying the local potentials of ν_*K near each point in $\nu(\text{Exc}(\nu))$ by taking the regularized maximum of the local potential and $|z|^2 - C$ in a local chart (C is a sufficiently large constant), see e.g. [11, Lemma 3.1] or [23, Remark 3.5].

Since $p^*(f^*[\gamma] - [\alpha]) = 0$, and p^* is injective, we conclude that $f^*[\gamma] = [\alpha]$, i.e. (1.3) holds. In other words, we have obtained that

$$-\lambda c_1(K_X) = [\omega_0] - f^*[\gamma],$$

which implies that $-K_X$ is f -ample. \square

3. FINITE TIME EXTINCTION

In this section we prove Conjecture 1.3 when $n \leq 3$. As we remarked in the introduction, it suffices to show that (1.4) implies (1.5).

First, we make the following general observation.

Lemma 3.1. *(X^n, ω_0) be a compact Kähler manifold, let $\omega(t)$ be the solution of the Kähler-Ricci (1.1), defined on the maximal time interval $[0, T)$ with $T < \infty$, and such that*

$$(3.1) \quad \text{diam}(X, \omega(t)) \rightarrow 0,$$

as $t \rightarrow T$. Then we have that

$$(3.2) \quad \text{Vol}(X, \omega(t)) \rightarrow 0,$$

as well, so that the flow exhibits finite time collapsing.

Proof. As usual let $[\alpha] = [\omega_0] - 2\pi T c_1(X)$ be the limiting class along the flow. If we had

$$\int_X \alpha^n > 0,$$

then [11, Theorem 1.5] shows that on the Zariski open set $X \setminus \text{Null}(\alpha)$ we have smooth convergence of $\omega(t)$ to a limiting Kähler metric ω_T on this set. In particular, the diameter of $(X, \omega(t))$ cannot go to zero. \square

From now on we assume that (1.4) holds. In particular, thanks to [11, Proposition 4.2] (or Lemma 2.4), we have that K_X is not pseudoeffective. Of course the content of Conjecture 1.3 is to show that the limiting class α is zero.

The following is the main result of this section:

Theorem 3.2. *If Conjecture 1.1 holds for X , then Conjecture 1.3 holds as well.*

In particular, combining this with Theorem 2.3, Corollary 2.6 and Theorem 2.7, we obtain the proof of Theorem 1.4. We also see that Conjecture 1.3 holds under the same hypotheses of Proposition 2.5 (a fact which was already mentioned in [35, Remark 1.1]). The proof of this theorem is a modification of the arguments in [35].

Proof. Thanks to Lemma 3.1 we are in the setup of Conjecture 1.1, and so we have a Fano fibration $f : X \rightarrow Y$ such that $[\omega_0] = -\lambda c_1(K_X) + f^*[\omega_Y]$, for some $\lambda > 0$ and some Kähler metric ω_Y on Y (in the sense of analytic spaces). The maximal existence time for the flow is thus $T = \frac{\lambda}{2\pi}$, and the limiting class is $[\alpha] = f^*[\omega_Y]$. Then $f^*\omega_Y$ is a smooth semipositive $(1, 1)$ form on X (this follows easily from the definition of Kähler metrics and holomorphic maps between analytic spaces), in the limiting class $[\alpha]$. We then write

$$\hat{\omega}_t = \frac{1}{T}((T-t)\omega_0 + t f^*\omega_Y),$$

which are Kähler metrics for all $0 \leq t < T$, and we have

$$\omega(t) = \frac{1}{T}((T-t)\omega_0 + t f^*\omega_Y) + \sqrt{-1}\partial\bar{\partial}\varphi(t),$$

where $\varphi(t)$ solves the parabolic complex Monge-Ampère equation

$$(3.3) \quad \begin{cases} \frac{\partial}{\partial t}\varphi(t) = \log \frac{(\hat{\omega}_t + \sqrt{-1}\partial\bar{\partial}\varphi(t))^n}{\Omega} \\ \varphi(0) = 0 \end{cases}$$

and Ω is a smooth positive volume form with $\sqrt{-1}\partial\bar{\partial}\log \Omega = \frac{1}{T}(f^*\omega_Y - \omega_0)$. A simple maximum principle argument gives $|\varphi(t)| \leq C$ on $X \times [0, T)$. We now want to use the usual Schwarz Lemma argument to show that on $X \times [0, T)$ we have

$$(3.4) \quad \omega(t) \geq C^{-1} f^*\omega_Y.$$

To prove this, we first claim that at every point where $\text{tr}_{\omega(t)}(f^*\omega_Y) > 0$ we have

$$(3.5) \quad \left(\frac{\partial}{\partial t} - \Delta \right) \log \text{tr}_{\omega(t)}(f^*\omega_Y) \leq C \text{tr}_{\omega(t)}(f^*\omega_Y).$$

To see this, recall that by definition of a Kähler metric on a compact Kähler space, given every point $y \in Y$ there exists an open neighborhood U of y in Y with an embedding $U \hookrightarrow \mathbb{C}^N$, and a smooth and strictly plurisubharmonic function η on \mathbb{C}^N , such that ω_Y equals the restriction of $\sqrt{-1}\partial\bar{\partial}\eta$ to U . Clearly if y is a smooth point of Y this just says that ω_Y is a usual Kähler metric near y . Therefore if $x \in X$ is a point with $\text{tr}_{\omega(t)}(f^*\omega_Y)(x) > 0$ and $f(x)$ is a smooth point of Y , then (3.5) holds near x thanks to a standard ‘‘Schwarz Lemma’’ calculation (see e.g. [41, Theorem 3.2.6]).

If on the other hand we have $\text{tr}_{\omega(t)}(f^*\omega_Y)(x) > 0$ but $f(x)$ is a singular point of Y , then we choose a neighborhood U of $f(x)$ as above, so that ω_Y equals the restriction of $\sqrt{-1}\partial\bar{\partial}\eta$ to U . On $f^{-1}(U)$ we then have the composite holomorphic map $\tilde{f} : f^{-1}(U) \rightarrow \mathbb{C}^N$ of f and the local embedding, such that on $f^{-1}(U)$ we have $f^*\omega_Y = \tilde{f}^*\sqrt{-1}\partial\bar{\partial}\eta$. Then we can apply the same Schwarz Lemma calculation as in [41, Theorem 3.2.6], to the holomorphic map between Kähler manifolds $\tilde{f} : (f^{-1}(U), \omega(t)) \rightarrow (\mathbb{C}^N, \sqrt{-1}\partial\bar{\partial}\eta)$, and (3.5) then holds on $f^{-1}(U)$.

On the other hand we also have

$$\left(\frac{\partial}{\partial t} - \Delta \right) \varphi(t) = \dot{\varphi}(t) - n + \text{tr}_{\omega(t)}\hat{\omega}_t \geq \log \frac{\omega(t)^n}{\Omega} - n + \frac{1}{4}\text{tr}_{\omega(t)}(f^*\omega_Y) + \frac{1}{2}\text{tr}_{\omega(t)}\hat{\omega}_t,$$

provided that t is sufficiently close to T , which we may always assume. Therefore, if we choose A large enough, we have that

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \Delta \right) (\log \text{tr}_{\omega(t)}(f^*\omega_Y) - A\varphi(t)) \\ \leq -\text{tr}_{\omega(t)}(f^*\omega_Y) - \text{tr}_{\omega(t)}\hat{\omega}_t - A \log \frac{\omega(t)^n}{\Omega} + An, \end{aligned}$$

at all points where $\text{tr}_{\omega(t)}(f^*\omega_Y) > 0$. Therefore

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \Delta \right) \left(\log \text{tr}_{\omega(t)}(f^*\omega_Y) - A\varphi(t) - An(T-t)(\log(T-t)-1) \right) \\ \leq -\text{tr}_{\omega(t)}(f^*\omega_Y) - \text{tr}_{\omega(t)}\hat{\omega}_t - A \log \frac{\omega(t)^n}{(T-t)^n\Omega} + An, \end{aligned}$$

which combined with

$$\text{tr}_{\omega(t)}\hat{\omega}_t \geq \frac{T-t}{T} \text{tr}_{\omega(t)}\omega_0 \geq C^{-1} \left(\frac{(T-t)^n\Omega}{\omega(t)^n} \right)^{\frac{1}{n}} \geq A \log \frac{(T-t)^n\Omega}{\omega(t)^n} - C,$$

gives

$$\begin{aligned} & \left(\frac{\partial}{\partial t} - \Delta \right) \left(\log \operatorname{tr}_{\omega(t)}(f^* \omega_Y) - A\varphi(t) - An(T-t)(\log(T-t) - 1) \right) \\ & \leq -\operatorname{tr}_{\omega(t)}(f^* \omega_Y) + C, \end{aligned}$$

and a simple application of the maximum principle then gives (3.4).

Lastly, using (3.4) we can use the same argument as in [35, Theorem 4.1] and conclude that if B is a geodesic ball of ω_Y contained in the regular part of Y then the diameter of $f^{-1}(B)$ (as a subset of X) with respect to $\omega(t)$ is bounded below by a small multiple of the diameter of B with respect to ω_Y , and hence remains bounded uniformly away from zero as $t \rightarrow T$, a contradiction to our assumption (1.4). \square

To close, we make two simple observations.

Remark 3.3. Assuming that (1.4) holds, then the main result of [25] shows that $H^1(X, \mathbb{R}) = 0$. This of course is consistent with Conjecture 1.3, since Fano manifolds have vanishing first Betti number.

Remark 3.4. To prove Conjecture 1.3 in general, it would be enough to show that if (1.4) holds then the L^1 -type norm

$$\int_X |\omega(t)|_{\omega_0} \omega_0^n,$$

or the equivalent quantity

$$\int_X \omega(t) \wedge \omega_0^{n-1},$$

goes to zero as $t \rightarrow T$. Indeed, any of these would imply that

$$\int_X \alpha \wedge \omega_0^{n-1} = 0,$$

and the Khovanskii-Teissier inequality for nef classes (see e.g. [17])

$$\int_X \alpha \wedge \omega_0^{n-1} \geq \left(\int_X \alpha^2 \wedge \omega_0^{n-2} \right)^{\frac{1}{2}} \left(\int_X \omega_0^n \right)^{\frac{1}{2}},$$

implies that $\int_X \alpha^2 \wedge \omega_0^{n-2} = 0$. The result now follows from the Hodge-Riemann bilinear relations on Kähler manifolds, proved in [13]. Indeed, following their notation, we set $\omega_1 = \dots = \omega_{n-1} := \omega_0$, so that the condition $\int_X \alpha \wedge \omega_0^{n-1} = 0$ says that $\alpha \in P^{1,1}(X)$, while the condition $\int_X \alpha^2 \wedge \omega_0^{n-2} = 0$ says that $Q(\alpha, \alpha) = 0$. Since by [13, Theorem A] the bilinear form Q is positive definite on $P^{1,1}(X)$, this implies that $\alpha = 0$, as required.

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